

Wilson Action of Lattice Gauge Fields with An Additional Term from Noncommutative Geometry

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Abstract

Differential structure of lattices can be defined if the lattices are treated as models of noncommutative geometry. The detailed construction consists of specifying a generalized Dirac operator and a wedge product. Gauge potential and field strength tensor can be defined based on this differential structure. When an inner product is specified for differential forms, classical action can be deduced for lattice gauge fields. Besides the familiar Wilson action being recovered, an additional term, related to the non-unitarity of link variables and loops spanning no area, emerges.

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I Introduction

Our work gains inspiration from three sources. First, exterior algebra, the foundation of differential forms on differentiable manifolds, has an intimate relation with Clifford algebra, the root of spinor [1]. Moreover, exterior differential together with metric on differential manifolds can be realized strikingly on the spinor bundles by Dirac operator under A. Connes' operator theoretic construction of classical geometry [2]. Physically speaking the same results, fermions provide representations for differential structures as well as metric structures of manifolds [3][4]. Connes generalized this scenario to noncommutative algebra settings and introduced noncommutative geometry(NCG) [5]. Second, a lattice provides a marvelous model for NCG developed by Dimakis and Müller-Hoissen(D&M) [6]. D&M geometry on discrete sets is essentially cohomologic description of broken lines; there is no non-commutativity at the level of 0-step broken lines which are just functions on these sets, while the non-commutativity is characterized by that the ordering to combine two broken lines not all of 0-step can not be exchanged. Third, a proper lattice Dirac operator, with or without root in NCG, more or less being attempted to solve the problem of chiral fermion in lattice field theory(LFT), was pursued by different authors [7][8][9].

We devised a Dirac operator \mathcal{D} on a n -dimensional lattice recently with the following properties: i) Adopting *Connes' distance formula*, we prove that the *induced metric of \mathcal{D}* recovers Euclidean geometry on this lattice, when $n = 1, 2$ [10]; ii) \mathcal{D} is a *Fredholm operator of index zero*[11], i.e. an operator subjected to a “*geometric square-root*” condition $\mathcal{D}^2 \sim \mathbf{1}$ [12]; iii) A differential structure on the lattice defined by *reduction of calculus* within the formalism of D&M can be realized onto spinor space upon this lattice through \mathcal{D} , what's more this representation is *Junk-free* [13](see [14] for a mathematical reference); iv) Under a specific matrix representation, \mathcal{D} possesses a *staggered Dirac operator* interpretation, if \mathcal{D} is slightly modified to fit a “*physical square-root*” condition $\mathcal{D}^2 = \Delta$ where Δ is the lattice laplacian [13]. Till now we just explored properties of this operator on a lattice without presenting of gauge fields. In this contribution, gauge coupling is added within the conventional approach in Connes' geometry. We will show that the unitarity of representation of gauge groups results from the compatibility of involution of covariant differential and gauge transformations. The classical action of lattice gauge fields is calculated. Resulting from the differential structure defined by us, an additional term, which will vanish under the usual assumption of unitarity of link variables in LFT, appears aside to the familiar Wilson action for lattice gauge fields. The losing of unitarity of link variables

will be showed having effect on the induced metric on lattices. Other authors' works with similar purpose deserve to be referred here [15][16], though due to the difference of differential structure definitions they did not notice the additional term.

This paper is organized as following. In Section II, necessary notations and formalism of differential forms on a lattice are introduced for describing lattice gauge fields, and then field strength tensor is calculated. In Section III, a inner product on differential forms is defined and the classical action of lattice gauge fields, which differs from the conventional Wilson action by a term vanishing if link variables are unitary, is calculated. Finally we discuss the significance of non-unitary link variables and the mathematical rigidity of our work in Section IV.

II Differential Forms, Connection and Curvature on Lattices

Notations A n -dimensional lattice can be regarded as a discrete abelian group \mathcal{Z}_N^n , the direct product of n cyclic groups \mathcal{Z}_N , in which N is an integer being large enough. The elements in \mathcal{Z}_N can be written as $0, 1, \dots, N-1$ and the multiplication is just the addition modulo N . Let \mathcal{A} be the algebra of complex functions on \mathcal{Z}_N^n . A collection of delta functions on \mathcal{Z}_N^n is defined to be a subset of \mathcal{A} : $\epsilon^x(y) = \prod_{i=1}^n \delta_{y^i}^{x^i}, \forall x, y \in \mathcal{Z}_N^n$ where x^i, y^i are the i th components of x, y respectively. The algebraic structure of \mathcal{A} can be expressed as $\epsilon^x \epsilon^y = \prod_{i=1}^n \delta_{x^i}^{y^i}$, $\sum_x \epsilon^x = \mathbf{1}_{\mathcal{A}}$ where $\mathbf{1}_{\mathcal{A}}(x) = 1$, since all ϵ^x form a natural basis for \mathcal{A} . Shift operators acting on \mathcal{A} is defined by $(T_{\mu}^{\pm} f)(x) = f(x \pm \hat{\mu}), \forall x \in \mathcal{Z}_N^n, \forall f \in \mathcal{A}$ or equivalently $T_{\mu}^{\pm} \epsilon^x = \epsilon^{x \mp \hat{\mu}}$, where $\hat{\mu}$ denotes the unit vector along μ -axis. Introduce formal partial derivatives by $\partial_{\mu}^{\pm} := T_{\mu}^{\pm} - \mathbf{1}$ with $\mathbf{1}$ being the trivial action on \mathcal{A} . $End_{\mathcal{C}}(V)$ refers to \mathcal{C} -linear endomorphism algebra on a linear space V , thus $T_{\mu}^{\pm}, \mathbf{1}, \partial_{\mu}^{\pm}$ are all elements in $End_{\mathcal{C}}(\mathcal{A})$.

II.1 Differential Structure on Lattices

Based on the observation that $(\partial_{\mu}^{\pm} f)(x)$ are two completely independent numbers with x being fixed on a lattice, we introduce differential of a function f in \mathcal{A} in this way

$$df = \sum_{\mu=1}^n (\partial_{\mu}^{+} f \chi_{+}^{\mu} + \partial_{\mu}^{-} f \chi_{-}^{\mu})$$

in which χ_{\pm}^{μ} form a basis of *first-order differential forms*. This intuition gains its rigidity under the following construction. \mathcal{A} can be extended to be a graded differential algebra $(\Omega(\mathcal{A}), d)$ by the construction rules shown below: *i)* $\Omega(\mathcal{A}) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{A})$, $\Omega^0(\mathcal{A}) = \mathcal{A}$ and the elements in $\Omega^p(\mathcal{A})$ are referred as p-order differential forms or just p-forms; *ii)* $\Omega^p(\mathcal{A}) \cdot \Omega^q(\mathcal{A}) = \Omega^{p+q}(\mathcal{A})$; *iii)* $d : \Omega^k(\mathcal{A}) \rightarrow \Omega^{k+1}(\mathcal{A})$, $k = 0, 1, \dots$ is a linear map satisfying graded Leibnitz rule and nilpotent rule

$$d(\omega_p \omega') = d(\omega_p) \omega' + (-)^p \omega_p d(\omega'), \forall \omega_p \in \Omega^p(\mathcal{A}), \omega' \in \Omega(\mathcal{A}),$$

$$d^2 = 0;$$

iv) $\mathbf{1}_{\mathcal{A}}$ is the unit of $\Omega(\mathcal{A})$. To make the construction be consistent, some conditions on χ_{\pm}^{μ} have to be imposed,

$$\chi_{\pm}^{\mu} f = (T_{\mu}^{\pm} f) \chi_{\pm}^{\mu} \quad (1)$$

$$\{\chi^{\mu}, \chi^{\nu}\} = 0, \{\chi^{-\mu}, \chi^{-\nu}\} = 0, \{\chi^{\mu}, \chi^{-\nu}\} = \delta^{\mu\nu} d\chi^{\mu} = \delta^{\mu\nu} d\chi^{-\nu} \quad (2)$$

for all $\mu, \nu = 1, 2, \dots, n$. We refer [17][13] for a detailed account of these statements. Eq.(1) is the fundamental non-commutativity on lattices, which indicates that functions is no longer commutative with differential forms, while the “error” of this non-commutativity is a shift by one lattice spacing, hence vanishing under *continuum limit*. The geometric interpretation of above construction is that p-forms correspond to linear combinations of p-stepped broken lines on \mathcal{Z}_N^n . For example, let $n = 1$ and consider χ_+ . It is easy to show that $\chi_+ = \sum_{x=0}^{N-1} \epsilon^x d\epsilon^{x+1}$, thus χ_+ can be interpreted as combination of 1-step line segments from x to $x+1$ with coefficients all being one.

Next we formulate a noncommutative exterior differential algebra $(\Lambda(\mathcal{A}), d)$ by introducing an equivalent relation $d\chi_{\pm}^{\mu} \sim 0$ onto $(\Omega(\mathcal{A}), d)$, s.t. $\Lambda(\mathcal{A}) \cong \Omega(\mathcal{A}) / \sim$. This definition avoids the potential ambiguity of applying wedge product directly, which results from non-commutative relation Eq.(1). Equivalently and practically, $\Lambda(\mathcal{A})$ can be defined to be a subset of $\Omega(\mathcal{A})$ together with a projection Π such that $\Lambda(\mathcal{A}) = \Pi(\Omega(\mathcal{A}))$, in which Π is defined as

$$\Pi(f_0 \chi_s^{\mu} f_1 \chi_t^{\nu} f_3) := \frac{1}{2} f_0 (T_{\mu}^s f_1) (T_{\mu}^s T_{\nu}^t f_3) (\chi_s^{\mu} \chi_t^{\nu} - \chi_t^{\nu} \chi_s^{\mu}) = f_0 (T_{\mu}^s f_1) (T_{\mu}^s T_{\nu}^t f_3) \chi_s^{\mu} \wedge \chi_t^{\nu} \quad (3)$$

for all $f_{\alpha} \in \mathcal{A}$, $\alpha = 0, 1, 2$, $s, t \in \{+, -\}$, $\mu, \nu = 1, 2, \dots, n$. Note that $\Pi(f_0 \chi_s^{\mu} \cdot f_1 \chi_t^{\nu}) = f_0 (T_{\mu}^s f_1) \chi_s^{\mu} \wedge \chi_t^{\nu} \neq -(T_{\nu}^t f_0) f_1 \chi_s^{\mu} \wedge \chi_t^{\nu} = \Pi(f_1 \chi_t^{\nu} \cdot f_0 \chi_s^{\mu})$ generally, therefore $\omega_{(1)} \wedge \omega_{(1)}$ is not necessarily equal to zero for a

generic 1-form $\omega_{(1)}$. The notion of wedge product makes sense by the projection Π , and $(\Lambda(\mathcal{A}), d)$ will be referred as a differential structure on \mathcal{Z}_N^n .

II.2 Representation of Differential Structure, Dirac-Connes Operator on the Lattice

A “fermion” representation for $(\Lambda(\mathcal{A}), d)$ is developed in this subsection. Introduce a spinor space $\mathcal{H}_s = \mathcal{C}^{2^n}$, and let \mathcal{H} be a finite dimensional Hilbert space $\mathcal{A} \otimes \mathcal{H}_s$ under the conventional definition of inner product $(\psi_1, \psi_2) = \sum_{x \in \mathcal{Z}^n} \sum_{i=1}^{2^n} \overline{\psi_1^i(x)} \psi_2^i(x)$. \mathcal{A} is represented on \mathcal{H} by $\pi : \pi(f) = f \otimes \mathbf{1}_s$, thus \mathcal{H} is turned out to be a left free \mathcal{A} -module in mathematical literature. Now extend π to be the “fermion” representation on $End_{\mathcal{C}}(\mathcal{H})$ by specifying $\pi(\chi_{\pm}^{\mu})$ and applying algebraic homomorphism rule. First, define

$$\pi(\chi_{\pm}^{\mu}) = i\eta_{\pm}^{\mu}, \eta_{\pm}^{\mu} = T_{\mu}^{\pm} \otimes \Gamma_{\pm}^{\mu} \quad (4)$$

Γ_{\pm}^{μ} are generators of Clifford algebra of 2n-dimensional Euclidean space, thus satisfy that

$$\{\Gamma_{+}^{\mu}, \Gamma_{+}^{\nu}\} = 0, \{\Gamma_{-}^{\mu}, \Gamma_{-}^{\nu}\} = 0, \{\Gamma_{+}^{\mu}, \Gamma_{-}^{\nu}\} = \delta^{\mu\nu} \mathbf{1}_s, \mu, \nu = 1, 2, \dots, n$$

as well as that $(\Gamma_{\pm}^{\mu})^{\dagger} = \Gamma_{\mp}^{\mu}$. Accordingly,

$$\{\eta_{+}^{\mu}, \eta_{+}^{\nu}\} = 0, \{\eta_{-}^{\mu}, \eta_{-}^{\nu}\} = 0, \{\eta_{+}^{\mu}, \eta_{-}^{\nu}\} = \delta^{\mu\nu} \mathbf{1} \otimes \mathbf{1}_s, \mu, \nu = 1, 2, \dots, n$$

which is the representation of Eq.(2). Since $(T_{\mu}^{\pm})^{\dagger} = T_{\mu}^{\mp}$, there is $(\eta_{\pm}^{\mu})^{\dagger} = \eta_{\mp}^{\mu}$. Eq.(1) is realized as $\eta_{\pm}^{\mu} \pi(f) = \pi(T_{\mu}^{\pm} f) \eta_{\pm}^{\mu}$ and $\pi(df) = i[\mathcal{D}, \pi(f)]$ where

$$\mathcal{D} := \sum_{\mu=1}^n (\eta_{+}^{\mu} + \eta_{-}^{\mu})$$

\mathcal{D} is a so-called Dirac-Connes operator in NCG. One can check that the relation $\mathcal{D}^2 = n\mathbf{1} \otimes \mathbf{1}_s$ holds, which is the condition for a *Fredholm operator of index zero*. Consequently, $(\mathcal{H}, \mathcal{D})$ forms a *Fredholm module* on \mathcal{A} . Besides, it is obvious that \mathcal{D} is hermitian, $\mathcal{D}^{\dagger} = \mathcal{D}$. Second, Eq(3) is implemented by defining the product of two conjoint $\eta_s^{\mu}, \eta_t^{\nu}$ to be wedge product, thus a formal noncommutative \mathcal{A} -linear rule $\eta_s^{\mu} \wedge (\eta_t^{\nu} f) = \eta_s^{\mu} \wedge ((T_{\nu}^t f) \eta_t^{\nu}) = (\eta_s^{\mu} (T_{\nu}^t f)) \wedge \eta_t^{\nu} = ((T_{\mu}^s T_{\nu}^t f) \eta_s^{\mu}) \wedge \eta_t^{\nu}$ follows.

II.3 Gauge Fields on Lattices

Let \mathcal{C}^k be the color space \mathcal{H}_c upon which gauge group G is represented as $R : G \rightarrow Aut_{\mathcal{C}}(\mathcal{H}_c)$. Directly product \mathcal{H} with \mathcal{H}_c to form a new \mathcal{A} -module $\tilde{\mathcal{H}} = \mathcal{H}_c \otimes \mathcal{H}$. $\pi(\Lambda(\mathcal{A}))$ acts on $\tilde{\mathcal{H}}$ as $\mathbf{1}_c \otimes \omega, \forall \omega \in \pi(\Lambda(\mathcal{A}))$

to which we will still use the symbol ω . Let $\tilde{\Lambda}(\mathcal{A}) = \text{End}_{\mathcal{C}}(\mathcal{H}_c) \otimes \pi(\Lambda(\mathcal{A}))$ consisting of *differential forms valued in $\text{End}_{\mathcal{C}}(\mathcal{H}_c)$* which are still referred as differential forms without introducing confusion. A sequence of concepts in gauge theory can be developed following a conventional routine. Define *connection 1-form* to be

$$A = i \sum_{\mu=1}^n (A_{\mu}^{+} \eta_{+}^{\mu} + A_{\mu}^{-} \eta_{-}^{\mu}) \in \tilde{\Lambda}(\mathcal{A})$$

in which gauge potential A_{μ}^{\pm} act trivially on spinor space \mathcal{H}_s . A is required to be anti-hermitian, $A^{\dagger} = -A$, hence satisfying that $A_{\mu}^{-} = T_{\mu}^{-}(A_{\mu}^{+\dagger})$. Let *covariant differential* be $\nabla := i\mathcal{D} + A$, so $\nabla^{\dagger} = -\nabla$. Introduce *parallel transport* by $U_{\mu}^{\pm} = \mathbf{1} + A_{\mu}^{\pm}$, then

$$\nabla = i \sum_{\mu=1}^n (U_{\mu}^{+} \eta_{+}^{\mu} + U_{\mu}^{-} \eta_{-}^{\mu})$$

and

$$U_{\mu}^{-} = T_{\mu}^{-}(U_{\mu}^{+\dagger}) \quad (5)$$

Note importantly that U_{μ}^{\pm} is referred as link variables in physical literature and that, however, no unitarity as a prescription is forced on U_{μ}^{\pm} in this article. *Curvature* or *field strength tensor* is defined to be $F = -\nabla \wedge \nabla = -i\{\mathcal{D}, A\}_{\wedge} - A \wedge A$. It is easy to check that $F^{\dagger} = F$ and that *Bianchi identity* holds for F , namely $i[\nabla, F]_{\wedge} = 0$. Even if the gauge group is abelian, there is still an A^2 term in curvature due to non-commutativity. The detailed form of F can be computed using either gauge potentials A_{μ}^{\pm} or parallel transports U_{μ}^{\pm} . After a sheet of paper's algebra, one will reach that $F = \sum_{\mu\nu} (F_{\mu\nu}^{++} \eta_{+}^{\mu} \wedge \eta_{+}^{\nu} + F_{\mu\nu}^{+-} \eta_{+}^{\mu} \wedge \eta_{-}^{\nu} + F_{\mu\nu}^{-+} \eta_{-}^{\mu} \wedge \eta_{+}^{\nu} + F_{\mu\nu}^{--} \eta_{-}^{\mu} \wedge \eta_{-}^{\nu})$ in which

$$F_{\mu\nu}^{++} = \frac{1}{2}((\partial_{\mu}^{+} A_{\nu}^{+} - \partial_{\nu}^{+} A_{\mu}^{+}) + (A_{\mu}^{+}(T_{\mu}^{+} A_{\nu}^{+}) - A_{\nu}^{+}(T_{\nu}^{+} A_{\mu}^{+}))) = \frac{1}{2}(U_{\mu}^{+}(T_{\mu}^{+} U_{\nu}^{+}) - U_{\nu}^{+}(T_{\nu}^{+} U_{\mu}^{+})) \quad (6)$$

$$F_{\mu\nu}^{+-} = \frac{1}{2}((\partial_{\mu}^{-} A_{\nu}^{+} - \partial_{\nu}^{-} A_{\mu}^{+}) + (A_{\mu}^{+}(T_{\mu}^{+} A_{\nu}^{-}) - A_{\nu}^{-}(T_{\nu}^{-} A_{\mu}^{+}))) = \frac{1}{2}(U_{\mu}^{+}(T_{\mu}^{+} U_{\nu}^{-}) - U_{\nu}^{-}(T_{\nu}^{-} U_{\mu}^{+})) \quad (7)$$

and $F_{\mu\nu}^{--}$, $F_{\mu\nu}^{-+}$ can be given by \leftrightarrow in $F_{\mu\nu}^{++}$, $F_{\mu\nu}^{+-}$. We point out that the exchange of \pm is a symmetry in our formalism which we will use broadly below, and that $F_{\mu\nu}^{++} = -F_{\nu\mu}^{++}$, $F_{\mu\nu}^{--} = -F_{\nu\mu}^{--}$, $F_{\mu\nu}^{-+} = -F_{\nu\mu}^{-+}$.

Gauge transformations are 0-forms valued in $R(G)$, denoted by g . If connect 1-form transforms affinely as $A' = gAg^{-1} + ig[\mathcal{D}, g^{-1}]$, then covariant differential and curvature transform adjointly as $\nabla' = g\nabla g^{-1}$, $F' = gFg^{-1}$. Involution of covariant differential and curvature is compatible with any gauge transformation g ,

i.e. $\nabla'^\dagger = -\nabla'$, $F'^\dagger = F'$, if $[\nabla, g^\dagger g] = 0$. Consequently, $g^\dagger g$ equals to unit up to an overall scalar factor; R becomes a unitary representation if this factor equals to one. Nevertheless, in our understanding, the unitarity of representation R does not imply necessarily the unitarity of link variables U_μ^\pm .

III Inner Product on Differential Forms and Classical Action of Gauge Fields

An inner product of differential forms $(,) : \tilde{\Lambda}(\mathcal{A}) \otimes \tilde{\Lambda}(\mathcal{A}) \rightarrow \mathcal{C}$ has to be specified, such that classical action for gauge fields on this lattice can be written as

$$S[U] = \frac{1}{2(tr_s \mathbf{1}_s)}(F, F) = \frac{1}{2(tr_s \mathbf{1}_s)}(\nabla^2, \nabla^2) \quad (8)$$

First we introduce a *hermitian structure* $\langle , \rangle : \tilde{\Lambda}(\mathcal{A}) \otimes \tilde{\Lambda}(\mathcal{A}) \rightarrow End_{\mathcal{C}}(\mathcal{H}_c) \otimes End_{\mathcal{C}}(\mathcal{A})$, $\tilde{\omega} \otimes \tilde{\omega}' \mapsto tr_s(\tilde{\omega}^\dagger \tilde{\omega}')$ for all $\tilde{\omega}, \tilde{\omega}' \in \tilde{\Lambda}(\mathcal{A})$ where tr_s is trace on $End_{\mathcal{C}}(\mathcal{H}_s)$. Then $(\tilde{\omega}, \tilde{\omega}') := tr_c Sp \langle \tilde{\omega}, \tilde{\omega}' \rangle$ where Sp is the trace on $End_{\mathcal{C}}(\mathcal{A})$ and tr_c is the trace on $End_{\mathcal{C}}(\mathcal{H}_c)$. Gauge invariance is guaranteed naturally under this definition of $S[U]$. To simplify the calculation of $S[U]$, we rewrite F to be

$$F = \sum_{\mu\nu} (F_{\mu\nu}^{++} \eta_+^\mu \wedge \eta_+^\nu + F_{\mu\nu}^{--} \eta_-^\mu \wedge \eta_-^\nu + 2F_{\mu\nu}^{+-} \eta_+^\mu \wedge \eta_-^\nu)$$

Consider a metric tensor

$$g_{(2)} := \begin{pmatrix} \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle & \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle & \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle \\ \langle \eta_+^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle & \langle \eta_+^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle & \langle \eta_+^\mu \wedge \eta_-^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle \\ \langle \eta_-^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle & \langle \eta_-^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle & \langle \eta_-^\mu \wedge \eta_-^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle \end{pmatrix}$$

Due the $+\leftrightarrow-$ symmetry, $\langle \eta_-^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle = \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle$, $\langle \eta_-^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle = -\langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle$, $\langle \eta_-^\mu \wedge \eta_-^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle = \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle$, $\langle \eta_+^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle = -\langle \eta_+^\mu \wedge \eta_-^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle$. Therefore, only $\langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle$, $\langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle$, $\langle \eta_+^\mu \wedge \eta_+^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle$, $\langle \eta_+^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle$, $\langle \eta_+^\mu \wedge \eta_-^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle$ need to be computed. Some algebra is needed to show

$$\langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_+^\rho \rangle = \frac{1}{4}(tr_s \mathbf{1}_s)(\delta^{\mu\lambda}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\lambda})\mathbf{1}_c \otimes \mathbf{1}$$

$$\langle \eta_+^\mu \wedge \eta_-^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle = \frac{1}{4}(tr_s \mathbf{1}_s)\delta^{\mu\lambda}\delta^{\nu\rho}\mathbf{1}_c \otimes \mathbf{1}$$

$$\langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^\lambda \wedge \eta_-^\rho \rangle = \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle = \langle \eta_+^\mu \wedge \eta_-^\nu, \eta_-^\lambda \wedge \eta_-^\rho \rangle = 0$$

Consequently,

$$g_{(2)} = \frac{1}{4}(tr_s \mathbf{1}_s) \begin{pmatrix} (\delta^{\mu\lambda}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\lambda}) & 0 & 0 \\ 0 & \delta^{\mu\lambda}\delta^{\nu\rho} & 0 \\ 0 & 0 & (\delta^{\mu\lambda}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\lambda}) \end{pmatrix} \mathbf{1}_c \otimes \mathbf{1}$$

Now apply the result of $g_{(2)}$,

$$\begin{aligned} \langle F, F \rangle &= \sum_{\mu\nu\mu'\nu'} (T_\mu^- T_\nu^+ (F_{\mu\nu}^{+-\dagger} F_{\mu'\nu'}^{+-}) \langle \eta_+^\mu \wedge \eta_-^\nu, \eta_+^{\mu'} \wedge \eta_-^{\nu'} \rangle + \\ &T_\mu^- T_\nu^- (F_{\mu\nu}^{++\dagger} F_{\mu'\nu'}^{++}) \langle \eta_+^\mu \wedge \eta_+^\nu, \eta_+^{\mu'} \wedge \eta_+^{\nu'} \rangle + T_\mu^+ T_\nu^+ (F_{\mu\nu}^{--\dagger} F_{\mu'\nu'}^{--}) \langle \eta_-^\mu \wedge \eta_-^\nu, \eta_-^{\mu'} \wedge \eta_-^{\nu'} \rangle) \\ &= \frac{1}{2}(tr_s \mathbf{1}_s) \sum_{\mu\nu} (2T_\mu^- T_\nu^+ (F_{\mu\nu}^{+-\dagger} F_{\mu\nu}^{+-}) + T_\mu^- T_\nu^- (F_{\mu\nu}^{++\dagger} F_{\mu\nu}^{++}) + T_\mu^+ T_\nu^+ (F_{\mu\nu}^{--\dagger} F_{\mu\nu}^{--})) \end{aligned} \quad (9)$$

Substitute detailed expressions in Eqs.(6)(7) into (9), and notice anti-hermitian condition (5),

$$\langle F, F \rangle = W + (tr_s \mathbf{1}_s) S$$

in which the symbols are defined in the following way:

Wilson term

$$\begin{aligned} W &= -\frac{1}{4}(tr_s \mathbf{1}_s) \sum_{\mu \neq \nu} (\mathcal{F}_{\mu\nu}^{-+} + \mathcal{F}_{\mu\nu}^{+-} + \mathcal{F}_{\mu\nu}^{++} + \mathcal{F}_{\mu\nu}^{--}) \\ \mathcal{F}_{\mu\nu}^{\mp\pm} &= \frac{1}{2}(P_{\mu\nu}^{\mp\pm} + P_{\nu\mu}^{\pm\mp}) - \mathbf{1}_c \otimes \mathbf{1}, \mathcal{F}_{\mu\nu}^{\pm\pm} = \frac{1}{2}(P_{\mu\nu}^{\pm\pm} + P_{\nu\mu}^{\pm\pm}) - \mathbf{1}_c \otimes \mathbf{1}, \mu \neq \nu \\ P_{\mu\nu}^{-+} &= U_\mu^- (T_\mu^- U_\nu^+) (T_\mu^- T_\nu^+ U_\mu^+) (T_\nu^+ U_\nu^-), P_{\mu\nu}^{+-} = P_{\mu\nu}^{-+} (+ \leftrightarrow -), \mu \neq \nu \\ P_{\mu\nu}^{--} &= U_\mu^- (T_\mu^- U_\nu^-) (T_\mu^- T_\nu^- U_\mu^+) (T_\nu^- U_\nu^+), P_{\mu\nu}^{++} = P_{\mu\nu}^{--} (+ \leftrightarrow -), \mu \neq \nu \end{aligned}$$

and an additional term

$$\begin{aligned} S &= \frac{1}{8} \sum_{\mu \neq \nu} (S_{\mu\nu}^{-+} + S_{\nu\mu}^{-+} + S_{\mu\nu}^{+-} + S_{\nu\mu}^{+-} + S_{\mu\nu}^{++} + S_{\nu\mu}^{++} + S_{\mu\nu}^{--} + S_{\nu\mu}^{--}) + \frac{1}{4} \sum_{\mu} (S_{\mu}^{++} + S_{\mu}^{--} - S_{\mu}^{+-} - S_{\mu}^{-+}) \\ S_{\mu\nu}^{\mp\pm} &= \Pi_{\mu\nu}^{\mp\pm} - \mathbf{1}_c \otimes \mathbf{1}, S_{\mu\nu}^{\pm\pm} = \Pi_{\mu\nu}^{\pm\pm} - \mathbf{1}_c \otimes \mathbf{1}, \mu \neq \nu \\ \Pi_{\mu\nu}^{-+} &= U_\nu^+ (T_\nu^+ U_\mu^-) (T_\mu^- T_\nu^+ U_\mu^+) (T_\nu^+ U_\nu^-), \Pi_{\mu\nu}^{+-} = \Pi_{\mu\nu}^{-+} (+ \leftrightarrow -), \mu \neq \nu \\ \Pi_{\mu\nu}^{--} &= U_\nu^- (T_\nu^- U_\mu^-) (T_\mu^- T_\nu^- U_\mu^+) (T_\nu^- U_\nu^+), \Pi_{\mu\nu}^{++} = \Pi_{\mu\nu}^{--} (+ \leftrightarrow -), \mu \neq \nu \\ S_{\mu}^{\mp\pm} &= \Pi_{\mu}^{\mp\pm} - \mathbf{1}_c \otimes \mathbf{1}, S_{\mu}^{\pm\pm} = \Pi_{\mu}^{\pm\pm} - \mathbf{1}_c \otimes \mathbf{1} \end{aligned}$$

$$\Pi_{\mu}^{-+} = U_{\mu}^{-}(T_{\mu}^{-}U_{\mu}^{+})U_{\mu}^{+}(T_{\mu}^{+}U_{\mu}^{-}), \Pi_{\mu}^{+-} = \Pi_{\mu}^{-+} (+ \leftrightarrow -)$$

$$\Pi_{\mu}^{--} = U_{\mu}^{-}(T_{\mu}^{-}U_{\mu}^{+})U_{\mu}^{-}(T_{\mu}^{-}U_{\mu}^{+}), \Pi_{\mu}^{++} = \Pi_{\mu}^{--} (+ \leftrightarrow -)$$

The geometric interpretations of $P_{\mu\nu}^{st}(x)$, $\Pi_{\mu\nu}^{st}(x)$, $\Pi_{\mu}^{st}(x)$ are Wilson-loop operators of parallel transports illustrated as

$$P_{\mu\nu}^{\mp\pm}(x) : x \rightarrow (x \pm \hat{\nu}) \rightarrow (x \mp \hat{\mu} \pm \hat{\nu}) \rightarrow (x \mp \hat{\mu}) \rightarrow x$$

$$P_{\mu\nu}^{\pm\pm}(x) : x \rightarrow (x \pm \hat{\nu}) \rightarrow (x \pm \hat{\mu} \pm \hat{\nu}) \rightarrow (x \pm \hat{\mu}) \rightarrow x$$

$$\Pi_{\mu\nu}^{\mp\pm}(x) : x \rightarrow (x \pm \hat{\nu}) \rightarrow (x \mp \hat{\mu} \pm \hat{\nu}) \rightarrow (x \pm \hat{\nu}) \rightarrow x$$

$$\Pi_{\mu\nu}^{\pm\pm}(x) : x \rightarrow (x \pm \hat{\nu}) \rightarrow (x \pm \hat{\mu} \pm \hat{\nu}) \rightarrow (x \pm \hat{\nu}) \rightarrow x$$

where $\mu \neq \nu$, and

$$\Pi_{\mu}^{\mp\pm}(x) : x \rightarrow (x \pm \hat{\mu}) \rightarrow x \rightarrow (x \mp \hat{\mu}) \rightarrow x$$

$$\Pi_{\mu}^{\pm\pm}(x) : x \rightarrow (x \pm \hat{\mu}) \rightarrow x \rightarrow (x \pm \hat{\mu}) \rightarrow x$$

The Wilson-loops producing $P_{\mu\nu}^{st}(x)$ span *fundamental plaquettes*, while those producing $\Pi_{\mu\nu}^{st}(x)$ and $\Pi_{\mu}^{st}(x)$ span no area. Involutive properties can be checked as

$$(P_{\mu\nu}^{\mp\pm})^{\dagger} = P_{\nu\mu}^{\pm\mp}, (\Pi_{\mu\nu}^{\pm\pm})^{\dagger} = \Pi_{\nu\mu}^{\pm\pm}, \mu \neq \nu$$

$$(\Pi_{\mu\nu}^{\mp\pm})^{\dagger} = \Pi_{\mu\nu}^{\mp\pm}, (\Pi_{\mu\nu}^{\pm\pm})^{\dagger} = \Pi_{\mu\nu}^{\pm\pm}, \mu \neq \nu$$

$$(\Pi_{\mu}^{\mp\pm})^{\dagger} = \Pi_{\mu}^{\pm\mp}, (\Pi_{\mu}^{\pm\pm})^{\dagger} = \Pi_{\mu}^{\pm\pm}$$

Collect these results and calculate $S[U]$ in Eq.(8) giving

$$S[U] = S_W[U] + S_{NU}[U]$$

S_W is of the form of standard Wilson action for lattice gauge fields in LFT up to a normalization factor

$$S_W[U] = - \sum_{f.p.} tr_c \mathcal{F}(f.p.)$$

where f.p. refers to fundamental plaquette and $\mathcal{F}(f.p.)$ equals to one $\mathcal{F}_{\mu\nu}^{st}(x)$ whose $(-)^s \hat{\mu}, (-)^t \hat{\nu}$ span this fundamental plaquette; while,

$$S_{NU}[U] = \frac{1}{2} Sp(tr_c S) = \frac{1}{2} \sum_{x \in \mathcal{Z}^n} (tr_c S)(x)$$

and will vanish if parallel transport U_{μ}^{\pm} satisfy unitarity $(U_{\mu}^{\pm})^{\dagger} U_{\mu}^{\pm} = \mathbf{1}_c \otimes \mathbf{1}$.

IV Discussions

IV.1 Non-unitary Link Variables

After the tedious calculation in the last section, we show that an additional term gaining contributions from those Wilson-loops spanning no area has to be added to the classical action of gauge fields on the lattice, if no unitarity of link variables is assumed. Here we illustrate the geometric significance of non-unitary parallel transports on Connes' distance by an one-dimensional example. Connes' distance on \mathcal{Z}^n can be defined by $d_{\mathcal{D}}(x, y) = \sup\{|f(x) - f(y)| : \|\mathcal{D}, \pi(f)\| \leq 1\}, \forall x, y \in \mathcal{Z}^n$. Let $n = 1, k = 1$, then \mathcal{D}, ∇ take on the forms like

$$\mathcal{D}_{n=1} = \begin{pmatrix} 0 & T^+ \\ T^- & 0 \end{pmatrix}, \nabla_{n=1} = i \begin{pmatrix} 0 & U^+ T^+ \\ U^- T^- & 0 \end{pmatrix}$$

In [18][10], it is showed that $d_{\mathcal{D}_{n=1}}(x, y) = |x - y|$. Now consider $d_{(-i)\nabla_{n=1}}(x, y)$, and one can check that $\|(-i)\nabla, \pi(f)\| = \sup\{|U^+|^2(x)|\partial^+ f|^2(x) : x \in \mathcal{Z}\}$, noticing the anti-hermitian condition Eq.(5); therefore, if U^+ is unitary, $d_{(-i)\nabla_{n=1}} = d_{\mathcal{D}_{n=1}}$, else then

$$d_{(-i)\nabla_{n=1}}(x, x+k) = \frac{1}{|U^+(x)|} + \frac{1}{|U^+(x+1)|} + \dots + \frac{1}{|U^+(x+k-1)|}$$

for all $x \in \mathcal{Z}, k = 1, 2, \dots$, namely non-unitary link variables will modify induced metric on lattices.

IV.2 Mathematical Rigidity

Till now we do not apply Connes' formalism in a restrictive manner. In fact in [2], Rennie showed that what are recovered from Connes' axioms for commutative algebras are necessarily $spin^C$ -manifolds. Hence, a lattice is outside to be a rigid model of Connes' formalism being applied to commutative algebras, unless additional structures like a *real structure*[19] or equivalently a charge conjugation in physics jargons is considered. As one aspect of this contradiction in our understanding, most constructions of Dirac-Connes operators on discrete sets including ours fail to fulfill *first order axiom* which requires $[[\mathcal{D}, f], g] = 0, \forall f, g \in \mathcal{A}$. However, it can be shown that the "error" caused by this violation is proportional to the lattice spacing. In fact, introduce a to be lattice spacing and still consider the previous one-dimensional example with $T^\pm \rightarrow T^\pm/a$ in the expression of $\mathcal{D}_{n=1}$ and $\partial^\pm \rightarrow (T^\pm - 1)/a$, there is

$$[[\mathcal{D}, \pi(f)], \pi(f')] = a \begin{pmatrix} 0 & (\partial^+ f)(\partial^+ f')T^+ \\ (\partial^- f)(\partial^- f')T^- & 0 \end{pmatrix} \quad (10)$$

Under continuum limit, the should-be vanishing of Eq.(10) is restored. As a suggestion, we would like to modify *first order axiom on lattices* to be $[[\mathcal{D}, f], g] = \mathcal{O}(a), \forall f, g \in \mathcal{A}$, where \mathcal{O} measures the convergency of operators under continuum limit.

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